where $\bar{E}_i$ is the incident electric field in V·m^{-1}. The induced magnetic field in the resonator is, again at resonance [6],

$$\bar{H} = -\frac{\bar{P}_e \cdot \bar{p}_e}{|\bar{p}_e|^2}(B_{11})_{\rm norm} \sin\frac{\pi z}{L} R_1\left(x_{11}\frac{r}{a}\right)\bar{u}_\phi$$

$$= 3.96 \sin\frac{\pi z}{L} J_1\left(3.8317\frac{r}{a}\right)E_i\bar{u}_\phi$$

as $\bar{p}_e = -j(a^3/Nc)\alpha\bar{u}_z$. The value of α, read off on Fig. 6, is 3.14.

REFERENCES

[1] J. Van Bladel, "On the resonances of a dielectric resonator of very high permittivity," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-23, pp. 199–208. Feb. 1975.
[2] C. G. Montgomery, *Technique of Microwave Measurements.* New York: McGraw-Hill, 1947, p. 305.
[3] M. Abramowitz and I. Stegun, *Handbook of Mathematical Tables.* New York: Dover, 1965.
[4] For more details concerning numerical procedures, see M. Verplanken's Ph.D. thesis (in preparation).
[5] J. Westlake, *A Handbook of Numerical Matrix Inversion and Solution of Linear Equations.* New York: Wiley, 1968.
[6] J. Van Bladel, "The excitation of dielectric resonators of very high permittivity," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-23, pp. 208–217, Feb. 1975.

Resonant Frequency of a $TE_{01\delta}°$ Dielectric Resonator

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***Abstract*—The resonant frequency of a $TE_{01\delta}°$ dielectric resonator was obtained by assuming a cylindrical surface containing the circumference of a dielectric resonator as a magnetic wall. In such a method, the error was less than 10 percent. In this short paper, the resonant frequency is obtained by a variational method, where the surface impedance is variated from a infinite value [1]. The theoretical value of a resonant frequency has a good agreement with our experimental result with an error less than 1 percent.**

I. INTRODUCTION

In the past, the resonant frequency of a dielectric resonator was obtained by assuming a cylindrical surface containing the circumference of a dielectric resonator as a magnetic wall [2], and with this method error was less than 10 percent between the theoretical values and the experimental ones.

Yee [3] obtained the modified open-circuit boundary (OCB) approximation method for $TE_{lm0}°$ mode but he assumed a cylindrical surface as a magnetic wall, and he obtained the variational method for TM_{1mn} mode, but he assumed two flat surfaces as a magnetic wall.

In our method, however, we do not assume only a cylindrical surface but also two flat surfaces as a magnetic wall. We assume the exponential decay in the z direction, and variate the surface impedance from a infinite value.

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In the case where the sectional areas of a cylindrical dielectric resonator face the metal plates at a short distance as shown in Fig. 1(a) (as in the case of microwave integrated circuit (MIC) use, for example), the assumption of a magnetic wall mentioned before is quite reasonable because the RF field of the resonator is in parallel to the metal plates and crosses the cylindrical surface almost vertically. However, in the case where the resonator is placed in a free space or the distance between the resonator and the metal plates becomes large, the parallel component of the RF field approaches the axis of a resonator. This does not satisfy a magnetic wall assumption. The situation is shown in Fig. 1(b).

II. THEORETICAL VALUES BY MAGNETIC WALL APPROXIMATION

When assuming the surface S_0 to be a magnetic wall as shown in Fig. 2, the resonant frequency can be obtained from (1)

$$k_z \cdot \tan(k_z l/2) = \alpha \quad J_0(k_\rho a) = 0$$

$$k_\rho^2 + k_z^2 = \epsilon_r k_0^2 \quad k_\rho^2 - \alpha^2 = k_0^2$$

$$k_0 = \omega_0(\epsilon_0\mu_0)^{1/2} \quad x_0 = 2\pi a/\lambda \quad \xi = l/2a \tag{1}$$

where k_z is a propagation constant in a dielectric region along the z axis, α is a damping constant in an air region inside a space surrounded by S_0 along the axis, k_ρ is a wavenumber along a radius, and k_0 is a free-space wavenumber.

III. THEORETICAL VALUES OBTAINED BY A VARIATIONAL METHOD

We assume the trial scalar functions internal to S_0 to be ϕ_{1d} and ϕ_{1a}

$$\phi_{1d} = J_0(k_\rho \rho)\cos(k_z z) \qquad \text{(in dielectric)}$$

$$\phi_{1a} = \exp(\alpha l/2)\cos[k_z(l/2)]J_0(k_\rho\rho)\exp(-\alpha|z|) \qquad \text{(in air)}. \tag{2}$$

In this case, the wall admittance Y_ρ can be expressed by (3) using the scalar function of (2)

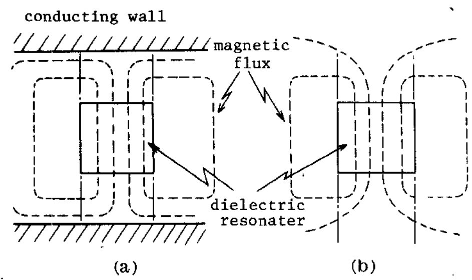

Fig. 1. Magnetic flux for $TE_{01\delta}°$ mode. (a) Near the conducting wall. (b) In free space.

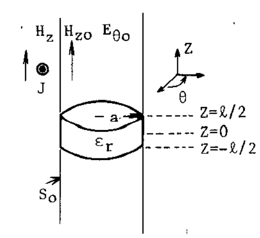

Fig. 2. Cylindrical coordinate system for dielectric cylinder.

$$Y_\rho = \left(\frac{H_z}{E_\theta}\right)_{\rho=a} = \left(\frac{k_\rho^2}{j\omega\mu_0}\phi \Big/ \frac{\partial\phi}{\partial\rho}\right)_{\rho=a}. \quad (3)$$

In (1) and (2), ω takes the values ω_0 when the surface S_0 is a magnetic wall, that is, $\Delta Y_\rho = 0$. However, Y_ρ takes a finite value ΔY_ρ in a practical case; and k_ρ, k_z, and α should change their values to $k_\rho' = k_\rho + \Delta k_\rho$, $k_z' = k_z + \Delta k_z$, $\alpha' = \alpha + \Delta\alpha$. Also, ω_0 changes the values to $\omega_0' = \omega_0 + \Delta\omega$.

From (1), we can obtain the relation of (4) and (5)

$$\eta = \frac{\partial\alpha}{\partial k_z} = \frac{\alpha}{k_z} + \frac{k_z l}{2}\cdot\left(\frac{\alpha}{k_z}\right)^2 + \frac{k_z l}{2} \quad (4)$$

$$\frac{\partial k_\rho}{\partial\omega} = \frac{\omega\epsilon_0\mu_0(\epsilon_r\alpha\eta + k_z)}{k_\rho(\alpha\eta + k_z)}. \quad (5)$$

Substituting (2), (4), and (5) into (3), we get (6)

$$\Delta\omega = \frac{\alpha\eta + k_z}{\alpha\eta\epsilon_r + k_z}\cdot\frac{j}{a\epsilon_0}\cdot\Delta Y_\rho. \quad (6)$$

On the other hand, the RF field outside the surface S_0 can be obtained. The z component (which is in parallel to the axis of a cylindrical resonator) of the RF field H_z should satisfy the stationary formula of (7), [1]

$$\iint_{S_0} \boldsymbol{J}\cdot\boldsymbol{u}_\theta E_{\theta 0}\, dS = \iint_{S_0} (H_z - H_{z0})\boldsymbol{u}_z \times \boldsymbol{n}\cdot\boldsymbol{u}_\theta E_{\theta 0}\, dS = 0 \quad (7)$$

where $\boldsymbol{J}$ is a trial current on the surface S_0 to satisfy the boundary conditions; $\boldsymbol{u}_z$ and $\boldsymbol{u}_\theta$ are unit vectors toward the z and θ directions, respectively; $\boldsymbol{n}$ is a unit vector perpendicular to the surface S_0; $E_{\theta 0}$ and H_{z0} are θ and z components of the electric field and magnetic field inside S_0.

In (7), $H_{z0} = 0$ in the case of $\Delta Y_\rho = 0$, and H_{z0} of (7) takes the values of (8) for the case of $\Delta Y_\rho \neq 0$

$$H_{z0} = \Delta Y_\rho E_{\theta 0}. \quad (8)$$

Substituting (8) into (7), we get (9)

$$\Delta Y_\rho = \frac{\iint_{S_0} H_z E_{\theta 0}\, dS}{\iint_{S_0} E_{\theta 0}^2\, dS}. \quad (9)$$

In the case of $\epsilon_r \gg 1$, we get (10) from (6) and (9)

$$\frac{\Delta\omega}{\omega_0} = \frac{j(k_z + \alpha\eta)}{a\alpha\eta\omega_0\epsilon_0\epsilon_r}\cdot\frac{\iint_{S_0} H_z E_{\theta 0}\, dS}{\iint_{S_0} E_{\theta 0}^2\, dS}$$

$$= \frac{j(k_z + \alpha\eta)}{a\alpha\eta\omega_0\epsilon_0\epsilon_r}\cdot\frac{\int_{-\infty}^{\infty} H_z E_{\theta 0}\, dS}{\int_{-\infty}^{\infty} E_{\theta 0}^2\, dS}. \quad (10)$$

Putting the scalar function of the field outside the surface S_0 as $\phi_2(\rho,z)$ and its Fourier transform as $\bar{\Phi}_2(\rho,w)$, we get the relation of (11)

$$\bar{\Phi}_2(\rho,w) = \int_{-\infty}^{\infty} \phi_2(\rho,z)\exp(-jwz)\, dz$$

$$\phi_2(\rho,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\Phi}_2(\rho,w)\exp(jwz)\, dw. \quad (11)$$

$\bar{\Phi}_2$ can be also expressed by (12), [4]

$$\bar{\Phi}_2(\rho,w) = f(w)H_0^{(2)}[\rho(k_0^2 - w^2)^{1/2}]$$

$$f(w) = \frac{1}{(k_0^2 - w^2)^{1/2}H_0^{(2)\prime}[a(k_0^2 - w^2)^{1/2}]}\int_{-\infty}^{\infty} E_{\theta 0}(a,z')\cdot\exp(-jwz')\, dz'. \quad (12)$$

$\phi_2(\rho,z)$ can be obtained from (11) and (12), and we have the relationship of (13) from which we get (14)

$$\boldsymbol{H} = -j\omega\epsilon_0\boldsymbol{u}_z\phi_2 + \frac{1}{j\omega\mu_0}\nabla\nabla\cdot\boldsymbol{u}_z\phi_2 \quad (13)$$

$$H_z = \frac{j}{2\pi\omega_0\mu_0}\int_{-\infty}^{\infty}(k_0^2 - w^2)^{1/2}\cdot\frac{H_0^{(2)}[a(k_0^2 - w^2)^{1/2}]}{H_1^{(2)}[a(k_0^2 - w^2)^{1/2}]}\exp(jwz)\, dw \cdot\int_{-\infty}^{\infty} E_{\theta 0}(a,z')\exp(-jwz')\, dz'. \quad (14)$$

We can get $E_{\theta 0}$ from the relation of $E_{\theta 0} = \partial\phi_1/\partial\rho$ by using ϕ_1 in (2). Substituting the values of $E_{\theta 0}$ into (14) and (10), we get $\Delta\omega/\omega_0$ as in (15)

$$\frac{\Delta\omega}{\omega_0} = F(x_z,x_\rho,x_\alpha,x_0,\xi,\epsilon_r)\cdot$$

$$= \frac{-2(x_z + x_\alpha\eta)\int_{-\infty}^{\infty}(x_0^2 - x^2)^{1/2}\frac{[H_0^{(2)}(x_0^2 - x^2)^{1/2}]}{[H_1^{(2)}(x_0^2 - x^2)^{1/2}]}A^2\, dx}{\pi x_\alpha x_0^2\epsilon_r\eta\left[\xi + \frac{\sin(2x_z\xi)}{2x_z} + \frac{\cos^2(x_z\xi)}{x_\alpha}\right]}$$

$$A = \frac{x_z\sin(x_z\xi)\cdot\cos(x\xi)}{x_z^2 - x^2} + \frac{x_\alpha\cos(x\xi)\cdot\cos(x_z\xi)}{x_\alpha^2 + x^2} - x\sin(x\xi)\cdot\cos(x_z\xi)\cdot\frac{x_z^2 + x_\alpha^2}{(x_z^2 - x^2)(x_\alpha^2 + x^2)} \quad (15)$$

where $x_z = k_z a$, $x_\rho = k_\rho a$, $x_0 = k_0 a$, $x_\alpha = wa$, $\xi = l/2a$. Since we get the values of x_z, x_ρ, x_α, and x_0 from ξ in (1), we get the relation of (16)

$$\frac{\Delta\omega}{\omega_0} = F(\xi,\epsilon_r). \quad (16)$$

$\Delta\omega/\omega_0$ can be obtained from (16) by using ξ and ϵ_r.

IV. COMPARISON BETWEEN THEORETICAL VALUES AND EXPERIMENTAL ONES

In the case of $\epsilon_r = 35$ and 88, the values of resonant frequencies computed from (1) (magnetic wall approximation) are shown by a dashed line, and the values computed from (15) (variational method) are shown by a solid line. The experimental values are shown with signs (×) in Fig. 3. Fig. 3 shows that the approximate values obtained from (1) are smaller than the experimental values by about 10 percent. On the contrary, the values obtained by a

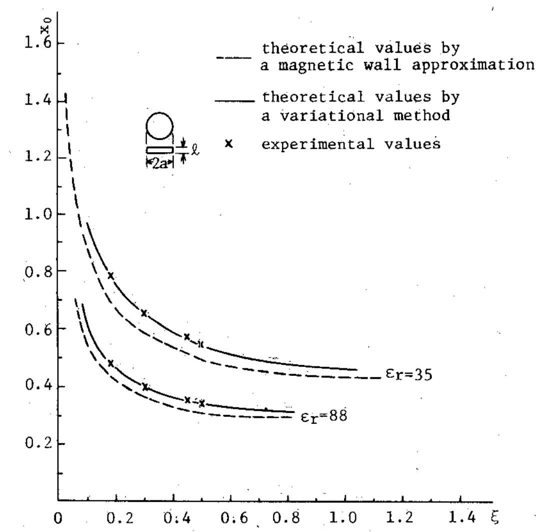

Fig. 3 Comparison between theoretical values and experimental ones.

variational method agree well with the experimental ones to within an error of 1 percent.

REFERENCES

[1] Y. Konishi and N. Hoshino, in *1971 Nat. Conv. Rec. Inst. Electron. Commun. Eng. Jap.*, p. 515.

[2] S. B. Cohn, "Microwave bandpass filters containing high-Q dielectric resonators," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-16, pp. 218–227, Apr. 1968.

[3] H. Y. Yee, "Natural resonance frequency of microwave dielectric resonator," *IEEE Trans. Microwave Theory Tech.* (Corresp.), vol. MTT-13, p. 256, Mar. 1965.

[4] R. F. Harrington, *Time-Harmonic Electromagnetic Fields.* New York: McGraw-Hill, 1961.

Reciprocity Theorem for a Region with Inhomogeneous Bianisotropic Media and Surface Impedance

MASANORI KOBAYASHI

Abstract—It is shown that the modified reciprocity theorem holds for a region bounded by inhomogeneous anisotropic impedance surfaces and composed of regions with lossless inhomogeous bianisotropic media; and besides, the reciprocity theorem holds for the case with a condition.

I. INTRODUCTION

Bianisotropic media or moving media have been treated in the literature [1]–[13], [18]. A moving medium, even if it is isotropic in its rest frame, must be treated as bianisotropic [6]. Recently, it was shown by Kong and Cheng [1] that a properly modified reciprocity theorem could be applied to bianisotropic media; they discussed the modified reciprocity theorem between a single-bianisotropic medium region and its complementary region. These regions are either bounded by perfectly conducting surfaces, or by surfaces that recede to infinity. However, waveguides with surface impedance walls are also of interest [14], [15].

In this short paper, the modified reciprocity theorem is derived in a region bounded by impedance surfaces; and besides, the reciprocity theorem is derived for the case with a condition.

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II. THEORY

Consider a region R consisting of p subregions in a three-dimensional space. The region R is bounded by inhomogeneous anisotropic impedance surface S and is composed of the regions R_i ($i = 1,2,\cdots,p$). Let the region R_i be filled with a lossless inhomogeneous bianisotropic medium [3], [18] and be bounded by the surface S_i.

Maxwell's equations for a time-harmonic and finite electric current density, $\boldsymbol{J}(\boldsymbol{r}) \exp(j\omega t)$, in such a region R are given by

$$\nabla \times \boldsymbol{E}(\boldsymbol{r}) = -j\omega \boldsymbol{B}(\boldsymbol{r}) \tag{1}$$

$$\nabla \times \boldsymbol{H}(\boldsymbol{r}) = j\omega \boldsymbol{D}(\boldsymbol{r}) + \boldsymbol{J}(\boldsymbol{r}) \tag{2}$$

subject to the constitutive relations

$$\boldsymbol{D}(\boldsymbol{r}) = \bar{\bar{\varepsilon}}(\boldsymbol{r})\cdot\boldsymbol{E}(\boldsymbol{r}) + \bar{\bar{\kappa}}(\boldsymbol{r})\cdot\boldsymbol{H}(\boldsymbol{r}) \tag{3}$$

$$\boldsymbol{B}(\boldsymbol{r}) = \bar{\bar{\nu}}(\boldsymbol{r})\cdot\boldsymbol{E}(\boldsymbol{r}) + \bar{\bar{\mu}}(\boldsymbol{r})\cdot\boldsymbol{H}(\boldsymbol{r}) \tag{4}$$

and the impedance boundary condition

$$\hat{n} \times \boldsymbol{E}(\boldsymbol{r}) = -\hat{n} \times [\bar{\bar{Z}}(\boldsymbol{r})\cdot\{\hat{n} \times \boldsymbol{H}(\boldsymbol{r})\}] \quad \text{on } S \tag{5}$$

and the conditions at the interfaces $S_i \cap S_k$ ($i,k = 1,2,\cdots,p$; $i \neq k$)

$$\hat{n}_i \times \{\hat{n}_i \times \boldsymbol{E}(\boldsymbol{r})\}|_{S_i\cap S_k, r\in S_i} = \hat{n}_k \times \{\hat{n}_k \times \boldsymbol{E}(\boldsymbol{r})\}|_{S_k\cap S_i, r\in S_k} \tag{6}$$

$$\hat{n}_i \times \{\hat{n}_i \times \boldsymbol{H}(\boldsymbol{r})\}|_{S_i\cap S_k, r\in S_i} = \hat{n}_k \times \{\hat{n}_k \times \boldsymbol{H}(\boldsymbol{r})\}|_{S_k\cap S_i, r\in S_k} \tag{7}$$

where

$\bar{\bar{\varepsilon}}, \bar{\bar{\mu}}, \bar{\bar{\kappa}}, \bar{\bar{\nu}}$	tensors of rank 2;
$\bar{\bar{\varepsilon}} = \bar{\bar{\varepsilon}}_i,\ \bar{\bar{\mu}} = \bar{\bar{\mu}}_i,\ \bar{\bar{\kappa}} = \bar{\bar{\kappa}}_i,\ \bar{\bar{\nu}} = \bar{\bar{\nu}}_i$	when $\bar{r} \in R_i$ ($i = 1,2,\cdots,p$);
$\hat{n}, \hat{n}_i$	outer normal unit vectors to S and S_i ($i = 1,2,\cdots,p$), respectively;
$\bar{\bar{Z}}$	= surface impedance tensor.

Now let us assume that the medium is such that [3], [18]

$$\bar{\bar{\varepsilon}}^{*T} = \bar{\bar{\varepsilon}} \tag{8}$$

$$\bar{\bar{\mu}}^{*T} = \bar{\bar{\mu}} \tag{9}$$

$$\bar{\bar{\kappa}}^{*T} = \bar{\bar{\nu}} \tag{10}$$

and let the field solutions corresponding to current sources $\boldsymbol{J}_a$ and $\boldsymbol{J}_b$ be $\boldsymbol{E}_a$ and $\boldsymbol{H}_a$, and $\boldsymbol{E}_b$ and $\boldsymbol{H}_b$, respectively. We obtain

$$\nabla\cdot(\boldsymbol{E}_a^* \times \boldsymbol{H}_b + \boldsymbol{E}_b \times \boldsymbol{H}_a^*) = -\boldsymbol{J}_a^*\cdot\boldsymbol{E}_b - \boldsymbol{J}_b\cdot\boldsymbol{E}_a^* \tag{11}$$

where the asterisk denotes the complex conjugate and the superscript T denotes the transpose. Applying the divergence theorem to (11), we obtain the following modified Lorentz reciprocity theorem for the region R_i ($i = 1,2,\cdots,p$):

$$\iiint_{R_i} (-\boldsymbol{J}_a^*\cdot\boldsymbol{E}_b - \boldsymbol{J}_b\cdot\boldsymbol{E}_a^*)\, dV = \iint_{S_i} (\boldsymbol{E}_a^* \times \boldsymbol{H}_b + \boldsymbol{E}_b \times \boldsymbol{H}_a^*)\cdot\hat{n}_i\, dS. \tag{12}$$

By using (5), the integrand in the right-hand side of (12) becomes zero on the surface $S_i \cap S$ if the surface impedance tensor $\bar{\bar{Z}}(\boldsymbol{r})$, which is defined in (5), is such that